Question Sheet 1, Limits I. v1. 2020-21

1. When verifying the ε - δ definition of $\lim_{x\to a} f(x) = L$ you need to know the value of the limit, L, in advance. This question is about finding L. Without detailed proofs evaluate the following limits.

i) $\displaystyle\lim_{x\to 1} \frac{x^2 - x - 2}{x+1}$ ii) $\displaystyle\lim_{x\to -1} \frac{x^2 - x - 2}{x+1}$

iii) $\displaystyle\lim_{x\to 1} \left\{ \frac{1}{x-1} - \frac{2}{x^2-1} \right\}$ iv) $\displaystyle\lim_{t\to 9} \frac{9-t}{3-\sqrt{t}}$.

v) $\displaystyle\lim_{x\to 2} \frac{\frac{1}{2} - \frac{1}{2}}{x-2}$. vi) $\displaystyle\lim_{t\to 8} \frac{8-t}{2-\sqrt[3]{t}}$.

Hint: In part (iv) use the important identity

$$a - b = \left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right)$$

for all $a, b \geq 0$. This follows from the "**difference of squares**" formula

$$x^2 - y^2 = (x-y)(x+y)$$

with $a = x^2$ and $b = y^2$.

For part (vi) use a similar result based on

$$x^3 - y^3 = (x-y)\left(x^2 + xy + y^2\right).$$

2. Consider the following **Rough Work** when trying to verify the ε - δ definition of $\lim_{x\to 2} x^2 = 4$.

Assume $0 < |x-2| < \delta$. Consider

$$|f(x) - L| = \left|x^2 - 4\right| = |(x-2)(x+2)| < \delta\,|x+2|.$$

Assume $\delta \leq 1$ so $0 < |x-2| < \delta \leq 1$, i.e. $-1 < x-2 < 1$ and thus $3 < x+2 < 5$. For then

$$\left|x^2 - 4\right| < \delta\,|x+2| < 5\delta,$$

which we want $\leq \varepsilon$. Hence choose $\delta = \min(1, \varepsilon/5)$.

Question What do we get for δ if we replace the requirement $\delta \leq 1$ by

i) $\delta \leq 100$ or ii) $\delta \leq 1/100$?

Limits of Cubic Polynomials

In the next four questions we look at limits of cubic polynomials. There are so many questions because I want to highlight different aspects of the quadratic polynomial which arises.

3. i) Factorise $x^3 - 8$ into a linear and a quadratic factor.

   ii) Bound, from above,
$$\left|x^2 + 2x + 4\right|$$
   on the interval $1 < x < 3$.

   iii) Show that the ε - δ definition of
$$\lim_{x \to 2} x^3 = 8,$$
   is satisfied if we choose $\delta = \min(1, \varepsilon/19)$ given $\varepsilon > 0$.

4. Given $\varepsilon > 0$ find a $\delta > 0$ that verifies the ε - δ definition of
$$\lim_{x \to 3} x^3 = 27.$$

5. i) Factorise $x^3 - 6x - 4$.

   ii) Bound, from above, $|x^2 - 2x - 2|$ on the interval $|x + 2| < 1$.

   iii) Verify the ε - δ definition of
$$\lim_{x \to -2} \left(x^3 - 6x - 2\right) = 2,$$
   i.e. given $\varepsilon > 0$ find a $\delta > 0$ for which the definition is satisfied.

6. i) Factorise $x^3 - 4x^2 + 4x - 1$.

   ii) Bound from above $|x^2 - 3x + 1|$ on the interval $0 < |x - 1| < 1$.

   iii) Verify the ε - δ definition of
$$\lim_{x \to 1} \left(x^3 - 4x^2 + 4x + 1\right) = 2,$$
   i.e. given $\varepsilon > 0$ find a $\delta > 0$ for which the definition is satisfied.

Limits of Rational Functions

In the next two questions we take a result $\lim_{x\to a} f(x) = L$ and examine

$$\lim_{x\to a} \frac{f(x)-L}{x-a},$$

for this gives examples of limits of rational functions which are **not** defined at the limit point.

7. (Based on Question 3.iii). i) Calculate, without proof,

$$\lim_{x\to 2} \frac{x^3-8}{x-2}.$$

   ii) Fully factorise the polynomial

$$x^3-12x+16.$$

   iii) Prove the value found in Part i is correct by verifying the ε - δ definition of limit.

8. (Based on Question 5.) i) What is the value of

$$\lim_{x\to -2} \frac{x^3-6x-4}{x+2}\ ?$$

   ii) Prove your result by verifying the ε - δ definition of this limit.

In the previous two questions we have looked at the limits of rational functions at a point where the function is **not** defined. Now we look at examples where the rational function **is** well-defined at the limit point.

9. i) Show that

$$\frac{3}{4} < \frac{x+2}{x+3} < \frac{5}{6}$$

   for $1 < x < 3$.

   ii) Show that the ε - δ definition of

$$\lim_{x\to 2} \frac{x^2+2x+2}{x+3} = 2$$

   can be verified by the choice of $\delta = \min(1, 6\varepsilon/5)$.

10. Evaluate

$$\lim_{x \to 2} \frac{x^2 - 2x - 12}{x + 2}$$

and verify the ε - δ definition of the limit.

Finally

11. Why must any $\delta > 0$ used to verify the ε - δ definition of the limit of $\sqrt{x}$ as $x \to 9$ satisfy $\delta \leq 9$?

Given $\varepsilon > 0$ find a $\delta > 0$ for which the definition of

$$\lim_{x \to 9} \sqrt{x} = 3$$

is satisfied.

Hint Use the Hint to Question 1.